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THE ANNIHILATING-IDEAL GRAPH OF $\mathbb{Z}_n$ IS WEAKLY PERFECT

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Abstract. A graph is called weakly perfect if its vertex chromatic number equals its clique number. Let R be a commutative ring with identity and $\mathbb{A}(R)$ be the set of ideals with non-zero annihilator. The annihilating-ideal graph of R is defined as the graph $\mathbb{AG}(R)$ with the vertex set $\mathbb{A}(R)^* = \mathbb{A}(R) \setminus \{0\}$ and two distinct vertices I and J are adjacent if and only if $IJ = 0$. In this paper, we show that the graph $\mathbb{AG}(\mathbb{Z}_n)$, for every positive integer n, is weakly perfect. Moreover, the exact value of the clique number of $\mathbb{AG}(\mathbb{Z}_n)$ is given and it is proved that $\mathbb{AG}(\mathbb{Z}_n)$ is class 1 for every positive integer n.

1. Introduction

The study of algebraic structures using the properties of graphs became an exciting research topic in the past twenty years, leading to many fascinating results and questions. There are many papers on assigning a graph to a ring; for instance see [1], [2], [3], [6] and [7].

Throughout this paper we assume that all rings are commutative with identity. Furthermore, we denote the set of all ideals of a ring R by $\mathrm{I}(R)$.

We now recall some basic graph theoretic facts: Let G be a graph with the vertex set $V(G)$. For any $x \in V(G)$, $\deg_G(x)$ (or $\deg(x)$) represents the number of edges incident to x, called the *degree* of the vertex x in G. The maximum degree of vertices of G is denoted by $\Delta(G)$. A *clique* of G is a maximal complete subgraph of G and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the *clique number* of G. For a graph G, let $\chi(G)$ denote the *vertex chromatic number* of G, i.e., the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. Note that for every graph G, $\omega(G) \leq \chi(G)$. A graph G is said to be *weakly perfect* if $\omega(G) = \chi(G)$. Recall that a *k-edge coloring* of a graph G is an assignment of k colors $\{1, \ldots, k\}$ to the edges of G such that no two adjacent edges have

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the same color, and the *edge chromatic number* $\chi'(G)$ of a graph G is the smallest integer k such that G has a k-edge coloring. A graph G is called *class 1*, if $\chi'(G) = \Delta(G)$.

Let R be a ring. We call an ideal I of R, an *annihilating-ideal* if there exists a non-zero ideal J of R such that $IJ = 0$. We use the notation $\mathbb{A}(R)$ to denote the set of all annihilating-ideals of R. By the *annihilating-ideal graph* of R, denoted $\mathbb{AG}(R)$, we mean the graph with the vertex set $\mathbb{A}(R)^* = \mathbb{A}(R) \setminus \{0\}$ such that two distinct vertices I and J are adjacent if and only if $IJ = 0$. The annihilating-ideal graph was first introduced in [4], and some of the properties of the annihilating-ideal graph have been studied. In this article, we show that for every positive integer n, $\mathbb{AG}(\mathbb{Z}_n)$ is a weakly perfect class 1 graph.

2. Main Results

We start with the following: Let n be a natural number. Throughout the paper, without loss of generality, we assume that we are given prime factorization $n = p_1^{n_1} p_2^{n_2} \dots p_m^{n_m}$, where the p_i are pairwise distinct primes and the n_i are natural numbers such that $1 \leq n_1 \leq n_2 \leq \cdots \leq n_m$.

Remark 2.1. Consider $\mathbb{Z}_n$, the ring of integers modulo n. Then:

(i) $\mathbb{Z}_n$ is Artinian. Thus it follows from [4, Proposition 1.3] that every non-trivial ideal of $\mathbb{Z}_n$ is a vertex of $\mathbb{AG}(\mathbb{Z}_n)$.

(ii) It follows from Chinese Remainder Theorem that

$$\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{n_1}} \times \cdots \times \mathbb{Z}_{p_m^{n_m}}.$$

(iii) $I \in I(\mathbb{Z}_n)$ if and only if $I = I_1 \times \cdots \times I_m$, where $I_i \in I(\mathbb{Z}_{p_i^{n_i}})$.

(iv) It is not hard to see that $|\mathbb{A}(\mathbb{Z}_n)^*| = \prod_{i=1}^{m}(n_i + 1) - 2$.

Let $n = p_1^2 p_2^3$. The following example describes the structure of $\mathbb{AG}(\mathbb{Z}_n)$.

Example 2.2. Let $n = p_1^2 p_2^3$. Then $\mathbb{AG}(\mathbb{Z}_n)$ has the following properties:

(i) By Part (iv) of Remark 2.1,

$$|V(\mathbb{AG}(\mathbb{Z}_n))| = |\mathbb{A}(\mathbb{Z}_n)^*| = (3 \cdot 4) - 2 = 10.$$

Indeed, Part (i) of Remark 2.1 implies that

$$\begin{aligned} V(\mathbb{AG}(\mathbb{Z}_n)) = \big\{ &\langle p_1 \rangle \times 0, \langle p_1 \rangle \times \langle p_2 \rangle, \langle p_1 \rangle \times \langle p_2^2 \rangle, \langle p_1 \rangle \times \mathbb{Z}_{p_2^3}, 0 \times \langle p_2 \rangle, \\ &0 \times \langle p_2^2 \rangle, 0 \times \mathbb{Z}_{p_2^3}, \mathbb{Z}_{p_1^2} \times 0, \mathbb{Z}_{p_1^2} \times \langle p_2 \rangle, \mathbb{Z}_{p_1^2} \times \langle p_2^2 \rangle \big\}. \end{aligned}$$

(ii) It is not hard to check that the set

$$\mathcal{C} = \big\{ 0 \times \langle p_2^2 \rangle, \langle p_1 \rangle \times 0, \langle p_1 \rangle \times \langle p_2^2 \rangle, 0 \times \langle p_2 \rangle \big\}$$

is a clique of $\mathbb{AG}(\mathbb{Z}_n)$. In Theorem 2.4, we will prove that $\mathcal{C}$ is the maximal clique of $\mathbb{AG}(\mathbb{Z}_n)$ and $\omega(\mathbb{AG}(\mathbb{Z}_n)) = \chi(\mathbb{AG}(\mathbb{Z}_n)) = 4$.

(iii) Consider the vertices $\langle p_1\rangle \times 0$ and $0 \times \langle p_2^2\rangle$. One may check that

$$\begin{aligned}\Delta(\mathbb{AG}(\mathbb{Z}_n)) &= \deg(0 \times \langle p_2^2\rangle) = (3^2) - 1 - 1 = 7\\ &> \deg(\langle p_1\rangle \times 0) = (2 \cdot 4) - 1 - 1 = 6.\end{aligned}$$

To prove that $\mathbb{AG}(\mathbb{Z}_n)$ is weakly perfect we need the following lemma:

Lemma 2.3. *Let $m = 1$. Then*

$$\omega(\mathbb{AG}(\mathbb{Z}_n)) = \chi(\mathbb{AG}(\mathbb{Z}_n)) = \left\lceil \frac{n_1}{2} \right\rceil.$$

Proof. Let

$$\mathcal{C} = \left\{ \langle p_1^{n_1-1}\rangle, \ldots, \langle p_1^{\lfloor n_1/2 \rfloor}\rangle \right\}.$$

It is clear that $\mathcal{C}$ is a maximal clique of $\mathbb{AG}(\mathbb{Z}_n)$. To complete the proof, we color all vertices contained in $\mathcal{C}$ with different colors. Since the set of vertices not contained in $\mathcal{C}$ is an independent set of $\mathbb{AG}(\mathbb{Z}_n)$, we assign the color of the vertex $\langle p_1^{\lfloor n_1/2 \rfloor}\rangle$ to all vertices outside of $\mathcal{C}$. □

This lemma allows us to state and prove the next result:

Theorem 2.4. *The graph $\mathbb{AG}(\mathbb{Z}_n)$ is weakly perfect, for every positive integer n. Moreover,*

$$\omega(\mathbb{AG}(\mathbb{Z}_n)) = \prod_{i=1}^{m} t_i + k - 1,$$

where k is the number of odd n_i's and

$$t_i = \begin{cases} \lceil \frac{n_i}{2} \rceil & \text{if } n_i \text{ is odd,} \\ \frac{n_i}{2} + 1 & \text{if } n_i \text{ is even.} \end{cases}$$

Proof. Let

$$A_i = \left\{ 0, \langle p_i^{n_i-1}\rangle, \ldots, \left\langle p_i^{\lceil n_i/2 \rceil} \right\rangle \right\},$$

for $i = 1, \ldots, m$, and if n_j is odd define

$$I^j = 0 \times \cdots \times 0 \times \langle p_j^{\lfloor n_j/2 \rfloor}\rangle \times 0 \times \cdots \times 0.$$

Set $B = \{I^j \,|\, n_j \text{ odd}\}$ and

$$\mathcal{C} = \left(\left(\prod_{i=1}^{m} A_i \right) \cup B \right) \setminus \{0\}.$$

We claim that $\mathcal{C}$ is a clique of $\mathbb{AG}(\mathbb{Z}_n)$. Let $I = I_1 \times \cdots \times I_m$ and $J = J_1 \times \cdots \times J_m$ be two elements of $\mathcal{C}$ and suppose that $I, J \in \prod_{i=1}^{m} A_i$. By Lemma 2.3, $A_i \setminus \{0\}$ is a clique of $\mathbb{AG}(\mathbb{Z}_{p_i^{n_i}})$ for $i = 1, \ldots, m$. Thus $I_i J_i = 0$, for each $i = 1, \ldots, m$, implying $IJ = 0$. Now, with out loss of generality, assume that $I \in \prod_{i=1}^{m} A_i$ and $J \in B$. Then we have that

$$J = 0 \times \cdots \times 0 \times \langle p_j^{\lfloor n_j/2 \rfloor}\rangle \times 0 \times \cdots \times 0,$$

for some $1 \leq j \leq m$. Since n_j is odd, $\langle p_j^{\lfloor n_j/2 \rfloor}\rangle\langle p_j^{\alpha}\rangle = 0$, for every $\alpha \geq \lceil n_j/2 \rceil$, and hence $IJ = 0$. The case $I, J \in B$ is clear, as non-zero components of I and J appear in different places. Thus the claim is proved and hence $\omega(\mathbb{AG}(\mathbb{Z}_n)) \geq |\mathcal{C}|$.

We now show that $\mathcal{C}$ is a maximal clique of $\mathbb{AG}(\mathbb{Z}_n)$. Assume that $I = I_1 \times \cdots \times I_m$ is a vertex which is adjacent to every vertex contained in $\mathcal{C}$. Then I_i is adjacent to $\langle p_i^{\lfloor n_i/2 \rfloor}\rangle$ and hence $I_i = \langle p_i^{\alpha_i}\rangle$, where $\alpha_i \geq \lceil n_i/2 \rceil$, for $i = 1, \ldots, m$, implying $I \in \mathcal{C}$. Therefore, $\mathcal{C}$ is a maximal clique.

To complete the proof, it is enough to show that $\chi(\mathbb{AG}(\mathbb{Z}_n)) \leq |\mathcal{C}|$. First color all vertices contained in $\mathcal{C}$ with different colors. Now, let $I_1 \times \cdots \times I_m = I$ be a vertex not contained in $\mathcal{C}$. We continue the proof in the two following cases:

Case 1: I is not adjacent to at least one vertex in B:

Let $T = \{j \in \mathbb{N} \,|\, II^j \neq 0\}$ and assume that j_0 is the minimum element of T. Assign the color of the vertex I^{j_0} to I. We now show that if I and $J = J_1 \times \cdots \times J_m$ have the same color j_0, then they are not adjacent. However, if this is the case then clearly $II^{j_0} \neq 0$ and $JI^{j_0} \neq 0$. Thus $I_{j_0}J_{j_0} \neq 0$ and so $IJ \neq 0$, as desired.

Case 2: I is adjacent to every vertex in B:

Since I is adjacent to any vertex in B, there is at least one vertex in $\prod_{i=1}^{m} A_i$ that is not adjacent to I. We consider the vertex $K = K_1 \times \cdots \times K_m$, which is defined as follows: Set $K_i = 0$ if $I_i \in A_i \cup \langle p_j^{\lfloor n_j/2 \rfloor}\rangle$. If $I_i \notin (A_i \cup \langle p_j^{\lfloor n_j/2 \rfloor}\rangle) \setminus \{0\}$, then define $T_i = \{J \in A_i \setminus \{0\} \,|\, I_iJ \neq 0\}$ and set K_i to be the minimum element of T_i. It is easily seen that $K \in \mathcal{C}$. Assign the color of the vertex K to the vertex I. We will show that if I and $J = J_1 \times \cdots \times J_m \notin \mathcal{C}$ have the same color, then they are not adjacent. Assume that $K_i \neq 0$, for some $i, 1 \leq i \leq m$. Then $I_i, J_i \notin A_i \setminus \{0\}$ and hence $I_iJ_i \neq 0$. Therefore $IJ \neq 0$, as desired.

□

Theorem 2.4 leads to the following immediate corollary which shows that if $\mathbb{Z}_n$ is a direct product of m fields, then $\omega(\mathbb{AG}(\mathbb{Z}_n)) = m$.

Corollary 2.5. *If $n_1 = \cdots = n_m = 1$, then $\omega(\mathbb{AG}(\mathbb{Z}_n)) = m$.*

Proof. Using the same notation as Theorem 2.4, it is obvious that $k = m$. Since $n_1 = \cdots = n_m = 1$, we deduce that $t_1 = \cdots = t_m = 1$. By Theorem 2.4, $\omega(\mathbb{AG}(\mathbb{Z}_n)) = (1)^m + m - 1$, as desired. □

To prove that $\mathbb{AG}(\mathbb{Z}_n)$ is a class 1 graph, the following lemma is needed:

Lemma 2.6. ([5, Corollary 5.4]) *Let G be a simple graph. Suppose that for every vertex u of maximum degree, there exists an edge $\{u, v\}$ such that $\Delta(G) - deg(v) + 2$ is more than the number of vertices with maximum degree in G. Then $\chi'(G) = \Delta(G)$.*

We are now in a position to prove that $\mathbb{AG}(\mathbb{Z}_n)$ is class 1.

Theorem 2.7. *For every positive integer n the graph $\mathbb{AG}(\mathbb{Z}_n)$ is class 1.*

Proof. Suppose that $m = 1$ so that $n_1 > 1$. If $n_m \leq 3$, then there is nothing to prove, as $\mathbb{AG}(\mathbb{Z}_n)$ is a complete graph of order at most two. Thus we may assume that $n_m > 3$. However, then we have that $\langle p_m^{n_m - 1}\rangle$ is the only vertex which is adjacent to every other vertex and $\Delta(\mathbb{AG}(\mathbb{Z}_n)) = \deg(\langle p_m^{n_m-1}\rangle) = n_m - 2$. Since $\deg(\langle p_m\rangle) = 1$, we have $n_m - 2 - 1 + 2 > 1$; by Lemma 2.6, $\mathbb{AG}(\mathbb{Z}_n)$ is class 1.

Now suppose that $m > 1$ and let $n_1 = \cdots = n_m = 1$. If $m = 2$, then the result follows easily, so suppose that $m > 2$. It is not hard to see that if $u = 0 \times \cdots \times 0 \times \mathbb{Z}_{p_m}$ and $v = \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_{m-1}} \times 0$, then

$$\Delta(\mathbb{AG}(\mathbb{Z}_n)) = \deg(u) = 2^{m-1} - 1,$$

u is adjacent to v, $\deg(v) = 1$, and $\mathbb{AG}(\mathbb{Z}_n)$ has m vertices of maximum degree. Since $m > 2$, we have $2^{m-1} - 1 - 1 + 2 > m$; again by Lemma 2.6, $\mathbb{AG}(\mathbb{Z}_n)$ is class 1.

Suppose now that $n_m > 1$ and let $u = I_1 \times \cdots \times I_m$ be a vertex of maximum degree in $\mathbb{AG}(\mathbb{Z}_n)$, where $I_i \in I(\mathbb{Z}_{p_i^{n_i}})$. If $I_i = 0$, for some $1 \leq i \leq m$, then obviously I_i annihilates all ideals of $\mathbb{Z}_{p_i^{n_i}}$; however, we know that u is not zero. Thus there exists exactly one i, $1 \leq i \leq m$, such that $I_i \neq 0$. Since $\Delta(\mathbb{AG}(\mathbb{Z}_n)) = \deg(u)$, we derive that

$$\deg_{\mathbb{AG}\left(\mathbb{Z}_{p_i^{n_i}}\right)}(I_i) = \Delta\left(\mathbb{AG}\left(\mathbb{Z}_{p_i^{n_i}}\right)\right)$$

and hence $I_i = \langle p_i^{n_i - 1}\rangle$. Since $n_m \geq n_i$, for $i = 1, \ldots, m-1$, we deduce that $i = m$ and u is of the form $u = 0 \times \cdots \times 0 \times \langle p_m^{n_m - 1}\rangle$. Thus

$$\Delta(\mathbb{AG}(\mathbb{Z}_n)) = \deg(u) = \prod_{i=1}^{m-1}(t_i + 1)t_m - 1 - 1,$$

where $t_i + 1 = |\mathrm{I}(\mathbb{Z}_{p_i^{n_i}})|$. Note that the ideal 0 is not a vertex of $\mathbb{AG}(\mathbb{Z}_n)$ and u is not adjacent to itself. Consider the vertex

$$v = \mathbb{Z}_{p_1^{n_1}} \times \cdots \times \mathbb{Z}_{p_{m-1}^{n_{m-1}}} \times 0.$$

Then v is adjacent to u and $\deg(v) = t_m$ and so

$$\Delta(\mathbb{AG}(\mathbb{Z}_n)) - deg(v) + 2 = \prod_{i=1}^{m-1}(t_i + 1)(t_m) - t_m.$$

If $n_m > n_i$ for every $i = 1, \ldots, m-1$, then $\mathbb{AG}(\mathbb{Z}_n)$ has only one vertex of maximum degree. If $n_1 = \cdots = n_m$, then $\mathbb{AG}(\mathbb{Z}_n)$ has m vertices of maximum degree. In both cases, by an easy calculation, one can show that $\Delta(\mathbb{AG}(\mathbb{Z}_n)) - \deg(v) + 2$ is larger than the number of vertices with maximum degree in $\mathbb{AG}(\mathbb{Z}_n)$, as $m > 1$ and $n_m > 1$. The result now follows from Lemma 2.6. $\square$

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